JOURNAL OF COMPUTATIONAL PHYSICS 8, 156–161 (1971)

Note

On the Remainder of Truncated Asymptotic Series

Many functions widely used in physics are most easily calculable by using asymptotic series [1–4] for certain ranges of the argument. Since these series are divergent, one should use a limited number of terms to get the best evaluation; usually terms are taken down to the one of smallest absolute value. The possible error is then expected to be of the same magnitude as the last term. This result can be improved by taking only half of the last term, which was shown to make the possible error not larger than half of its previous value [1, 5]. Taking, for example, the asymptotic expansion of the exponential integral

$$E(x) = \int_x^\infty (e^{-t}/t)\, dt \sim (e^{-x}/x) \sum_{n=0} (-1/x)^n\, n!, \tag{1}$$

we have an approximation for the integral in the sense that while taking N terms, the error committed does not exceed the $(N+1)$th term. The smallest term is seen to occur when two consecutive terms have about the same value, which results in the condition $N \approx x$. For larger values of x, N increases accordingly and the possible error decreases very rapidly.

Airey [2] and Dingle [3] developed the method of "converging factor", which is a factor that should miltiply the last term taken in order to reduce the possible error substantially. In certain cases, however, the accuracy achieved by using the plain asymptotic series is sufficient while the simplicity of using this series is of advantage, especially for computer work. In these cases, it is of great importance to evaluate the possible error explicitly, without calculating the terms of the series. This has beeen done before for series representing the integrals appearing in glow curve theory [5, 6] and will be done here for various other asymptotic series. This evaluation, which also approximates the last term that is taken, is of importance when the accuracy needed is better than that given by the present method. If a converging factor is to be calculated, it is important to know to what accuracy it should be found, therefore an estimate of the last term taken is valuable.

One of the functions calculable by the aid of asymptotic series is the incomplete Γ-function $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1}\, dt$. The following treatment resembles that given before [5, 6] for the integrals appearing in glow curves theory, which are very close in nature to the incomplete gamma function. A special case for $a = 0$ is the exponential integral mentioned above. For $a > 1$, one can reduce the value of a to be between zero and unity by the recurrence equation

$$\Gamma(a, x) = e^{-x}x^{a-1} + (a-1)\Gamma(a-1, x). \tag{2}$$

The asymptotic series representing the function is a generalization of Eq. (1), namely,

$$\begin{aligned}\Gamma(a, x) &\sim x^{a-1}e^{-x} \sum_{n=0} (-1)^n\, \Gamma(n-a+1)/[x^n\Gamma(1-a)] \\ &= x^{a-1}e^{-x}[1 + (a-1)/x + (a-1)(a-2)/x^2 + \cdots],\end{aligned} \tag{3}$$

where $\Gamma(y)$ is the gamma function of y. The ratio between two consecutive terms N and $N+1$ is $(N-a+1)/x$. N should be chosen such that this ratio is the largest one smaller than unity; that is, $N - a + 1 \approx x$.

For best results, we have to take N terms in the series and the possible error is related to the $(N+1)$th term, a_{N+1}, which is $(-1)^{N+1}\Gamma(N-a+2)/[x^{N+1}\Gamma(1-a)]$. Defining $0 < \alpha < 1$ such that $N - a + 1 + \alpha = x$, we have

$$| a_{N+1} | = \Gamma(x+1-\alpha)/[x^{x+a-\alpha}\Gamma(1-a)]. \tag{4}$$

For values of x greater than or equal to 10, the Stirling approximation [7] is correct to 1 %; therefore, we have

$$\begin{aligned}| a_{N+1} | &\cong \sqrt{2\pi x}\, e^{-(x-\alpha)}(x-\alpha)^{x-\alpha+1/2}/[x^{x+a-\alpha}\Gamma(1-a)] \\ &= \sqrt{2\pi x}\, e^{-(x-\alpha)}(1-\alpha/x)^x\, (x-\alpha)^{-\alpha+1/2}/[x^{a-\alpha}\Gamma(1-a)].\end{aligned} \tag{5}$$

By writing $e^{-\alpha}$ instead of $(1-\alpha/x)^x$, we increase the expression by not more than 5 % since $\alpha < 1$ and $x \geqslant 10$. By taking $x^{-\alpha+1/2}$ instead of $(x-\alpha)^{-\alpha+1/2}$, we may add an error of up to 5 %. We finally have

$$| a_{N+1} | = \sqrt{2\pi x}\, e^{-x}x^{-a}/\Gamma(1-a). \tag{6}$$

The relative error is $| a_{N+1} |$ divided by the sum of the series. For $x \geqslant 10$, the sum is 0.9 or more; thus, taking $| a_{N+1} |$ itself as the relative error, decreases the estimation of the error by up to 10 %. Summing up all the approximations leading to Eq. (6), we have an expression for the relative error which may be wrong by up to about 10 % (taking into account that some of the factors mentioned are of opposite sign). The value of $\Gamma(1-a)$ for $1 > a \geqslant 0$ can be found in tables. The expression (6) becomes better for higher values of x. For the particular case of the exponential integral we have to insert $a = 0$ and get the simpler result $\sqrt{2\pi x}\, e^{-x}$.

As was shown before [5, 6], the value found by the series improves if we add $\frac{1}{2}a_{N+1}$ and the possible error reduces to $R_N = \frac{1}{2} \mid a_{N+1} \mid$, which was shown to reduce the expression in Eq. (6) by $\frac{1}{2}$ as well. Following Dingle [3], a better approximation for the series is

$$\sum_{n=0} \frac{\Gamma(n-a+1)}{(-x)^n \Gamma(1-a)} = \sum_{n=0}^{N-1} \frac{\Gamma(n-a+1)}{(-x)^n \Gamma(1-a)} + \frac{\Gamma(N-a+1)}{\Gamma(1-a)(-x)^N} \Lambda_{N-a}(x), \quad (7)$$

where $\Lambda_s(x)$ is a function calculable by a power series given by Dingle

$$\begin{aligned}\Lambda_s(s+\theta) = \tfrac{1}{2} - (1-8s)(1-2\theta) + (1/32s^2)(1-2\theta-4\theta^2) \\ + (1/128s^3)(1+6\theta+8\theta^2+8\theta^3) + \cdots, \quad (8)\end{aligned}$$

this expression is exact to order $1/s^3$. For $x = 10$, this means an accuracy of 10^{-3}. Checking this expression more closely, we see that for $s \geqslant 10$ and $\theta \leqslant 1$, namely for our case where $N - a + 1 \approx x \geqslant 10$ we have $0.49 \leqslant \Lambda_s(s+\theta) \leqslant 0.51$. This means that by the simple way of taking the terms down to the smallest but one and adding half of the smallest one, we get results which are now proved to be correct to $(1/100) \mid a_{N+1} \mid$ rather than $\frac{1}{2} \mid a_{N+1} \mid$. The estimate on the last term to be taken as in Eq. (6) should be combined with the desired accuracy in order to decide the number of terms to be taken in the expression (8).

The Stieltjes integral [1] $F(x) = \int_0^\infty e^{-t}\, dt/(t+x)$ is very similar to the exponential integral in the sense that its asymptotic expansion is

$$F(x) \sim \sum_{n=0} (-1)^n\, n!/x^{n+1} = (1/x) \sum_{n=0} (-1)^n\, n!/x^n \quad (9)$$

and its analysis will be the same. The error function Erfc $T = \int_T^\infty \exp(-u^2)\, du$ can be regarded a special case of the incomplete Γ function since Erfc $T = \frac{1}{2}\Gamma(\frac{1}{2}, T^2)$. Its asymptotic expansion is thus

$$\text{Erfc } T \sim (1/2T\sqrt{\pi}) \exp(-T^2) \sum_{n=0} \Gamma(n+\tfrac{1}{2})(-1)^n/T^{2n}. \quad (10)$$

The smallest term occurs for $T^2 \approx N + \frac{1}{2}$, thus, for $T > 3$, a good approximation for this term is $\sqrt{2}\exp(-T^2)$. As before, if we add only half of this term, the possible error is about $0.01\sqrt{2}\exp(-T^2)$.

Dealing with the Fresnel Integrals [3], one has the series

$$A(x) \sim \sum_{1,3,5\ldots} (-1)^{\frac{1}{2}(n-1)}\, \Gamma(n-\tfrac{1}{2})/[\Gamma(\tfrac{1}{2})\, x^{n-1}]; \quad (11)$$

The smallest term is found for $(N + \frac{1}{2})(N - \frac{1}{2}) \approx x^2$; for $x \approx 10$ or more we have $N \approx x$. The smallest term would now approximately be (in absolute value) $\Gamma(x + 3/2)/x^{x+1}$ which, by the use of Stirling's formula tends to $(2\pi/x)^{1/2}e^{-(x+1/2)}$. According to Dingle [3], this last term should be multiplied by $\Pi_{N+1/2}(x)$. The expansion of Π_s is given by

$$\Pi_s(s + \theta) = \tfrac{1}{2} - (\tfrac{1}{4}s)(1 - 2\theta) + (1/8s^2)(1 - 4\theta - 2\theta^2) - (1/16s^3)(3 + 18\theta + 8\theta^2) + \cdots, \tag{12}$$

accurate to about $1/s^3$. Again, s is about 10 or more and $0 < \theta < 1$; under these circumstances, $0.48 \leqslant \Pi_s(s + \theta) \leqslant 0.52$. Thus, adding one half of the Nth term brings about an error which does not exceed $0.02(2\pi/x)^{1/2}\, e^{-(x-1/2)}$, which is for $x \approx 10$ an accuracy of $\sim 10^{-6}$. The accuracy improves rapidly for larger values of x.

A similar case occurs with the sine and cosine integrals whose asymptotic series are

$$C(x) = \sum_{1,3,5\ldots}^{N \text{ odd}} \frac{(-1)^{\frac{1}{2}(n-1)}\,(n-1)!}{x^{n-1}} + \frac{(-1)^{\frac{1}{2}(N+1)}\,(N+1)!}{x^{N+1}}\,\Pi_{N+1}(x) \tag{13}$$

and a similar series with even values of n. The treatment is the same as in the previous case as is the approximation for the last term and the possible error which is up to 2 % thereof.

The possibility of taking $\frac{1}{2}$ as an approximate converging factor for the smallest term exists only for alternating sign series. When the terms in the asymptotic series have the same sign, the converging factors are of the form

$$\Lambda_s[-(s + \theta)] = -(1/3)(1 - 3\theta) + (1/135s)(4 - 45\theta + 90\theta^2 - 45\theta^3) + \cdots \tag{14}$$

and another series denoted $\Pi_s[i(s + \theta)]$; see Ref. [3]. These series may have various values even for large s and $0 < \theta < 1$.

Dawson's integral is given by

$$\operatorname{Erfi} x = \int_\infty^x \exp(u^2)\, du \sim [\exp(x^2)/2x] \sum_{n=0} \Gamma(n + \tfrac{1}{2})/[\Gamma(\tfrac{1}{2})\, x^{2n}]. \tag{15}$$

As with the error function, the smallest term is found for $x^2 \approx N + \frac{1}{2}$ and its value is approximated by $\sqrt{2}\exp(-x^2)$ for $x > 3$. This last term has to be multiplied by the converging factor and is now $\sqrt{2}\exp(-x^2)\,\Lambda_{N-1/2}(x^2)$, the possible error depends on the accuracy in determining Λ, namely on the number of terms taken in Eq. (14). The Goodwin–Staton integral is given by $f(x) = \int_0^\infty \exp(-u^2)/(u + x)\, du$ whose asymptotic expansion is

$$f(x) \sim (\tfrac{1}{2}x) \sum_{n=0} \Gamma(n + \tfrac{1}{2})/x^{2n}, \tag{16}$$

which is the same as Dawson's integral except for the factor $\Gamma(\frac{1}{2}) = \sqrt{\pi}$; the treatment would therefore be the same.

For calculating the Raabe integrals one has to use the asymptotic expansions

$$R_s(x) \sim (1/x) \sum_{1,3,5\ldots} (n-1)!/x^{n-1}, \tag{17}$$

and $R_c(x)$ which is the same for even values of n. These are the same as the functions $C(x)$ and $D(x)$ defined while dealing with the sine and cosine integrals, apart from the nonalternating sign of the terms here. In both cases, the smallest term will appear for $N \approx x$ and multiplied by the converging factor it gives

$$(2\pi x)^{1/2} e^{-x} \Pi_{N+1}(ix).$$

The Fermi–Dirac integrals are defined by

$$F_p(x) = (1/p!) \int_0^\infty \epsilon^p \, d\epsilon/(e^{\epsilon-x} + 1). \tag{18}$$

The asymptotic series for positive x is given here by

$$F_p(x) \sim \cos \pi p \cdot F_p(-x) + 2x^{p+1} \left\{ \sum_{0,2..}^{[p+1]_e} t(n)/[(p+1-n)!\, x^n] \right.$$

$$\left. + (\sin \pi p/\pi) \sum_{[p+3]_e}^{\infty} (n-p-2)!\, t(n)/x^n \right\}, \tag{19}$$

where $t(n) = \sum_{\nu=1} (-1)^{\nu-1}/\nu^n$, $t(0) = \frac{1}{2}$ and $F_p(-x)$ is given by a convergent series [3].

The ratio between two subsequent terms is

$$[(N-p)(N-p-1)/x^2][t(N+2)/t(N)]$$

which should be set to be about unity. For large enough x, we have

$$t(N+2) \approx t(N) \approx 1$$

and therefore $N - p - \frac{1}{2} \approx x$. By reasoning similar to the previous and using Eq. (17) in the third paper by Dingle [3] we have an expression for the last term to be taken

$$|\, a_N \,| \approx \sqrt{2\pi}\, x^{-(1+p)} e^{-x} \sum_{\nu=1}^{\infty} (-1)^{\nu-1}/\nu^{N+2} \Pi_{N-p}(i\nu x). \tag{20}$$

The possible error depends, again, on the accuracy of calculating $\overline{\Pi}$. A similar treatment can be given to the Bose–Einstein integrals

$$B_p(x) = (1/p!) \int_0^\infty \epsilon^p \, d\epsilon/(e^{\epsilon-x} - 1). \tag{21}$$

Acknowledgment

I would like to thank Mr. A Cappell for reading the manuscript and making some valuable comments.

References

1. T. J. Stieltjes, *Ann. Sci. Ecol. Norm. Sup.* **3** (1886), 201.
2. J. R. Airey, *Phil. Mag.* **24** (1937), 521.
3. R. B. Dingle, *Proc. Roy. Soc.* **A244** (1958), 456; **A249** (1959), 270.
4. E. T. Copson, "Asymptotic Expansions," the University Press, Cambridge, 1965.
5. R. Chen, *J. Compt. Phys.* **4** (1969), 415.
6. R. Chen, *J. Compt. Phys.* **6** (1970), 314.
7. G. H. Hardy, "Divergent Series," P. 333, Clarendon Press, Oxford, 1949.

Received: July 17, 1970

Reuven Chen

Department of Physics and Astronomy
Tel-Aviv University, Tel-Aviv, Israel